JOURNAL OF
SYNCHROTRON
RADIATION

ISSN: 1600-5775
journals.iucr.org/s

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J. Synchrotron Rad. (2017). **24**, 609–614

JOURNAL OF SYNCHROTRON RADIATION

ISSN 1600-5775

Effective aperture of X-ray compound refractive lenses

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Received 18 January 2017
Accepted 9 April 2017

Edited by A. Momose, Tohoku University, Japan

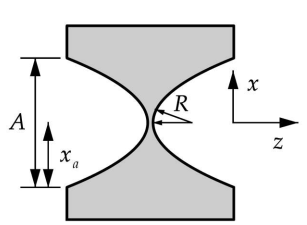
Keywords: X-ray focusing theory; compound refractive lens; effective aperture; X-ray absorption; computer simulations.

A new definition of the effective aperture of the X-ray compound refractive lens (CRL) is proposed. Both linear (one-dimensional) and circular (two-dimensional) CRLs are considered. It is shown that for a strongly absorbing CRL the real aperture does not influence the focusing properties and the effective aperture is determined by absorption. However, there are three ways to determine the effective aperture in terms of transparent CRLs. In the papers by Kohn [(2002). *JETP Lett.* **76**, 600–603; (2003). *J. Exp. Theor. Phys.* **97**, 204–215; (2009). *J. Surface Investig.* **3**, 358–364; (2012). *J. Synchrotron Rad.* **19**, 84–92; Kohn *et al.* (2003). *Opt. Commun.* **216**, 247–260; (2003). *J. Phys. IV Fr*, **104**, 217–220], the FWHM of the X-ray beam intensity just behind the CRL was used. In the papers by Lengeler *et al.* [(1999). *J. Synchrotron Rad.* **6**, 1153–1167; (1998). *J. Appl. Phys.* **84**, 5855–5861], the maximum intensity value at the focus was used. Numerically, these two definitions differ by 50%. The new definition is based on the integral intensity of the beam behind the CRL over the real aperture. The integral intensity is the most physical value and is independent of distance. The new definition gives a value that is greater than that of the Kohn definition by 6% and less than that of the Lengeler definition by 41%. A new approximation for the aperture function of a two-dimensional CRL is proposed which allows one to calculate the two-dimensional CRL through the one-dimensional CRL and to obtain an analytical solution for a complex system of many CRLs.

1. Introduction

The X-ray compound refractive lens (CRL) was successfully used for the first time with a synchrotron radiation source of the third generation (ESRF, Grenoble, France) by Snigirev *et al.* (1996). The linear (one-dimensional; 1D) CRL was created as an array of 30 cylindrical holes of 300 µm radius in an Al plate using a computer-controlled drilling machine. Later, the circular (two-dimensional; 2D) CRL with a parabolic surface profile was created by Lengeler *et al.* (1999). Pressing tools, consisting of two convex paraboloids with rotational symmetry facing each other and guided by a centering ring, were used. The aluminium plate in which the paraboloids were pressed from both sides is held and centered by a ring. Computer-controlled tooling machines allow the pressing tool to be manufactured with micrometer precision.

Another kind of CRL is the planar nanofocusing lens (Aristov *et al.*, 1999; Snigirev *et al.*, 2009; Schroer *et al.*, 2003). Such lenses are created inside the surface layer of a silicon crystal by means of microstructuring methods (electron-beam lithography and deep anisotropic etching) which are very well developed methods. Planar CRLs have an extremely short focal length (up to 1 cm), making it possible to reduce the beam in the focus down to sizes ten times smaller than a micrometer.



The theory of linear (1D) CRLs in the limit of strong absorption of radiation at the edge of the aperture was developed by Kohn in a series of papers (Kohn, 2002, 2003, 2009, 2012; Kohn *et al.*, 2003*a*,*b*). The main results of these works were an analytical solution of the problem for a very long CRL when the CRL length is comparable with or even greater than the focal length and recurrent relations which allow one to calculate a complex system of many CRLs.

The most complete theory was considered in the last paper (Kohn, 2012), hereon referred to as paper [1]. In this work the effective aperture of the CRL was determined as the FWHM (full width at half-maximum) of the radiation intensity profile just behind the CRL. The effective aperture of 2D CRLs was not considered because it was assumed to be the same (the diameter of the circular aperture). On the other hand, the effective aperture of 2D CRLs was considered in the works by Lengeler *et al.* (1998, 1999). The more complete calculation was performed in the Lengeler *et al.* (1999) paper, hereon referred to as paper [2].

The main goal of this work is to analyze why the formulae for the effective aperture proposed in papers [1] and [2] give different values under the same conditions, which is a problem because the results differ by 50%. The effective aperture is determined by comparing the formulae for the transparent CRL and the absorbing CRL. There are three channels for comparison: first, the maximum value of the intensity at the focus; second, the FWHM of the intensity profile; and, third, the integral intensity inside the real aperture. We have found that in paper [1] the second channel was used, but in paper [2] the first channel was used.

In this work we propose to use the third channel because the integral intensity does not change in air and it is the most physical parameter. We show that the value of such an effective aperture is 6% larger than the value given in paper [1] but 41% smaller than that given in paper [2]. We consider as well an approximation which allows one to calculate the intensity profile of a 2D CRL at any distance as a product of the 1D CRL intensity profiles over two coordinates x and y.

2. The linear (1D) X-ray CRL

Let us consider first the simpler case of a linear (1D) X-ray CRL. We assume that the CRL length is much shorter than the focal length of the CRL. In this case the CRL can be considered as a phase object which can be described by a standard phase contrast approximation. This means that the CRL can be described by means of the transmission function

$$T(x) = \exp\left[-i\pi(1-i\gamma)x^2/\lambda f\right], \qquad |x| < x_a, \tag{1}$$

where x is the coordinate across the beam (see Fig. 1), $x_a = A/2$, A is the aperture of the linear CRL, λ is the wavelength of monochromatic radiation, $f = R/2\delta N$ is the focal length of the CRL, $\gamma = \beta/\delta$, R is the curvature radius at the apex of the parabolic surface profile of a double concave lens, N is the number of such lenses in the CRL, and the complex refractive index of the CRL material is $n = 1 - \delta + i\beta$. If $|x| \geq x_a$, then $T(x) = T_a = T(x_a)$.

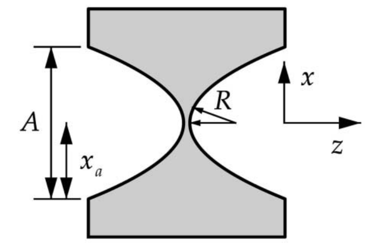


Figure 1
Parameters of the X-ray double concave lens and the coordinate axes.

We consider a simple experimental in-line setup where a source of synchrotron radiation is located at a distance z_0 in front of the CRL, and the intensity of radiation is detected at the distance z_1 behind the CRL. Then the intensity $I(x) = |E(x)|^2$ and

$$E(x) = \int dx_1\, P(x-x_1, z_1)\, T(x_1)\, P(x_1 - x_0, z_0), \tag{2}$$

where x_0 is the coordinate of a point in the transverse section of the source, and the function

$$P(x, z) = \frac{1}{(i\lambda z)^{1/2}} \exp\left[i\pi\left(x^2/\lambda z\right)\right] \tag{3}$$

is the Fresnel propagator.

Equation (2) can be written in another form taking into account equation (3),

$$E(x) = P(x-x_0, z_t) \int dx_1\, P(x_r - x_1, z_r)\, T(x_1), \tag{4}$$

where

$$z_r = \frac{z_1}{C_s}, \quad x_r = \frac{1}{C_s}\left(x + x_0\frac{z_1}{z_0}\right), \quad C_s = \frac{z_t}{z_0}, \quad z_t = z_0 + z_1. \tag{5}$$

We note that in equations (2) and (4) the aperture A restricts the region of integration where the function $T(x)$ is variable. Outside this region the transmission function is equal to a constant, T_a. Therefore, in the general case, we can write

$$E(x) = P(x-x_0, z_t)\left[T_a + \int_{-x_a}^{x_a} dx_1\; P(x_r - x_1, z_r)\left\{T(x_1) - T_a\right\}\right]. \tag{6}$$

Let us consider the case of a fully transparent CRL. It is a good approximation for visible light in glass, but it is a poor approximation for X-ray radiation in all materials. Then, $\beta = \gamma = 0$ and $|T(x)| = 1$. Two integrals in equation (6) can be presented analytically through special functions known as Fresnel integrals. However, for all distances which are less or greater than the focal length we can use the approximation of a stationary phase (SP) if $x_a^2 \gg \lambda f$. According to this approximation the main contribution to the integral is obtained from the region near the point SP where the first derivative of the phase equals zero. The integral near this point is calculated in the infinite limits but only under the condition that the point SP is inside the region of integration. In the opposite case, the point SP does not contribute to the integral.

Let us consider the second integral from the Fresnel propagator. It is evident that for each value of x_r there is only one point SP, namely $x_1 = x_r$ which gives 1 if $|x_r| < x_a$, and gives 0 in the opposite case. It follows from equation (5) that a shift x_0 of the point at the source leads to a shift of the total picture by the distance $-x_0 z_1/z_0$. This is why it is sufficient to consider the case when $x_0 = 0$ (the point source at the optical axis). Then the second integral can be calculated as $-T_a\theta(x_a C_s - |x|)$, where $\theta(x)$ is a step function which equals 1 for positive values of the argument and 0 for negative values. This term compensates the first term in the square brackets of equation (6) in the region where the size is slightly greater than the aperture if $z_0 \gg z_1$.

The first integral has only one point of SP inside the region of positive arguments of the function $\theta(x_a w - |x|)$, where $w = C_s(1 - z_r/f)$. The size of this region decreases linearly with increasing distance z_r from 0 to f. The square modulus of the contribution of this point SP to the integral is equal to $w^{-1}I_0$ where $I_0 = (\lambda z_0)^{-1}$ is the intensity of X-ray radiation at the CRL. We note that the multiplier C_s is obtained as a result of taking into account the first Fresnel propagator in equation (6).

As a result, we find that behind the CRL the part of the beam inside the aperture A with constant intensity I_0 is concentrated inside the region $wA < A$ with constant intensity $w^{-1}I_0 > I_0$. We see that this approximation allows a conservation of the integral intensity and corresponds to the geometrical optics, *i.e.* the CRL increases the concentration of rays inside the smaller region.

At the focal distance $z_r = f$ the SP approximation is not valid for the first integral, and we need a more accurate calculation. Fortunately, the result can be obtained analytically as follows,

$$I(x) = |E(x)|^2 = I_0 \frac{A}{\lambda z_1} \frac{\sin^2(\alpha x)}{(\alpha x)^2}, \quad \alpha = \frac{\pi A}{\lambda z_1}, \tag{7}$$

and the distance of the focused image of the source is equal to $z_1 = f(1 - f/z_0)^{-1}$.

We are interested in the maximum value of intensity I_m, the FWHM of the intensity profile w_m, and the integral intensity of the focal peak S. A direct calculation gives the equations

$$I_m = \frac{A^2}{\lambda z_1} I_0, \quad w_m = 0.8859 \frac{\lambda z_1}{A}, \quad S = AI_0. \tag{8}$$

Here we use the fact that the function $p(s) = \sin^2(s)/s^2$ equals 0.5 if $s = 1.3916$, and the integral of $p(s)$ equals π. Therefore, the accurate intensity profile at the distance of imaging the point source has the shape of a sharp peak with oscillating tails if $A \gg r_1 = (\lambda z_1)^{1/2}$. The parameter r_1 is called the radius of the first Fresnel zone because $P(r_1, z_1) = P(0, z_1)\exp(i\pi)$, *i.e.* the Fresnel propagator changes its sign at this distance.

For X-ray radiation the case considered above is impossible because all materials absorb the radiation and the refraction is small and therefore the curvature radius has to be small. Under the condition $A \gg r_1$ the thickness of the CRL matter at the aperture boundaries becomes large, and the absorption leads to a decreasing T_a. Let us consider the limit case when $T_a = 0$. Then the integral in (6) can be calculated in infinite limits and the real aperture A does not influence the result.

As a result we find that the intensity profile at the distance of the source image is described by a Gaussian function,

$$I(x) = \frac{1}{\gamma C_s} \exp\left(-\frac{2\pi}{\lambda z_1 \gamma C_s} x^2\right) I_0. \tag{9}$$

Now, instead of (8) we obtain the new values of parameters,

$$I_m = \frac{1}{\gamma C_s} I_0, \quad w_m = e_1 C_s (\lambda f \gamma)^{1/2}, \quad S = \left(\frac{\lambda f}{2\gamma}\right)^{1/2} I_0. \tag{10}$$

Here we used the relation $f = z_r = z_1/C_s$ and introduced the constant $e_1 = (2\ln 2/\pi)^{1/2} = 0.6643$.

We want to characterize the X-ray CRL by means of the parameter of the effective aperture A_e, which by definition has to be less than the real aperture. However, in this way we have several variants. In the paper [1] the effective aperture is defined as the FWHM w_0 of the intensity profile $|T(x)|^2 = \exp(-2\pi\gamma x^2/\lambda f)$ just behind the CRL. This definition gives a value $A_{e1} = w_0 = e_1(\lambda f/\gamma)^{1/2}$.

We note that this definition is not completely correct because the FWHM depends on distance. A more correct definition can be obtained by means of a comparison between the integral intensities for the transparent and absorbing CRLs. According to this definition, we obtain $A_e = (\lambda f/2\gamma)^{1/2}$. The main goal of this paper is to propose just this definition of the CRL effective aperture. We note that the new definition differs slightly from the definition for A_{e1} because $2^{-1/2}/e_1 = e_3$, where $e_3 = (4\ln 2/\pi)^{-1/2} = 1.0645$. Then $A_e = e_3 A_{e1}$. The difference is only 6%.

In paper [2] the effective aperture is defined from a comparison between the maximum values I_m for the transparent and absorbing CRLs. In this way we have $A_{e2} = (\lambda f/\gamma)^{1/2}$. Numerically, $A_{e2} = 1.4142 A_e = 1.505 A_{e1}$. This definition gives a value that is 41% greater than the new definition and 50% greater than the definition in paper [1].

We can write for the strongly absorbing CRL using the new definition of the effective aperture,

$$I_m = 2\frac{A_e^2}{\lambda z_1} I_0, \quad w_m = e_1^2 e_3 \frac{\lambda z_1}{A_e}, \quad S = A_e I_0. \tag{11}$$

We note that the constant $e_1^2 e_3 = 0.4697$. We see that the case of the absorbing CRL cannot be described completely by the equations of the transparent CRL. If the integral intensity coincides, then the maximum value results in a coefficient of 2, and the FWHM results in a coefficient that is slightly less than 0.5.

The case of the strongly absorbing CRL is of interest because the real aperture is not important and the integrals are calculated in the infinite limits. Then for the circular (2D) CRL the integrals over the second coordinate y have the same value and are calculated independently. The coordinates of the point source can be eliminated from the calculations because the total picture is shifted as a whole. Nevertheless, some differences exist and this is considered below.

In paper [2] the interpolation formula for the effective aperture of the circular (2D) CRL is derived in which both limit cases of the transparent and the strongly absorbing CRLs are involved as particular cases. To derive such formulae from the intensity maximum value at the focus it is necessary to calculate accurately the first integral in equation (6) at the central point $x_r = 0$. In the case of the linear (1D) CRL such an integral has no analytical expression, and the result can be written as

$$A_{e2} = \left(\frac{\lambda f}{\gamma}\right)^{1/2} \mathrm{erf}\left[\left(\frac{\pi\gamma}{\lambda f}\right)^{1/2} x_a\right], \qquad (12)$$

where $\mathrm{erf}(x)$ is the Gauss error function. $\mathrm{erf}(x) = 1$ for large values of the argument and we have $A_{e2} = (\lambda f/\gamma)^{1/2}$. For small values of the argument, $\mathrm{erf}(x) = 2\pi^{-1/2}x$, and we obtain the real aperture $A_{e2} = 2x_a = A$.

We can calculate the interpolation formula for the effective aperture from the integral intensity. Since the integral intensity is independent of distance, it is convenient to make a calculation for $z_1 = 0$, *i.e.* just behind the CRL. In such a way we obtain

$$A_e = 2\int_0^{x_a} \mathrm{d}x \left|T(x)\right|^2 = \left(\frac{\lambda f}{2\gamma}\right)^{1/2} \mathrm{erf}\left[\left(\frac{2\pi\gamma}{\lambda f}\right)^{1/2} x_a\right]. \qquad (13)$$

This new formula can be used for an estimation of the effective aperture of the linear (1D) CRL.

3. The circular (2D) X-ray CRL

The circular (2D) X-ray CRL has a circular aperture of radius R_a and area πR_a^2. We can characterize this aperture by it's diameter $D_a = 2R_a$. We have shown above that the focus peak is completely determined by the aperture. Therefore we are not interested in the region of integration outside the aperture. This allows us to restrict the area of integration by the aperture. In experiments, such a situation arises when the circular slit is used just in front of or behind the CRL.

In this case we have two transverse coordinate x, y and the radius $R = (x^2 + y^2)^{1/2}$. The transmission function can be written as

$$T(R) = \exp\left[-i\pi(1 - i\gamma)\,R^2/\lambda f\right], \quad R < R_a. \qquad (14)$$

To simplify the notation we introduce the 2D radius vector $\mathbf{R} = (x, y)$ with modulus R. Now the intensity of radiation is $I(\mathbf{R}) = \left|E(\mathbf{R})\right|^2$ and

$$E(\mathbf{R}) = \int \mathrm{d}\mathbf{R}_1\, P_2(\mathbf{R} - \mathbf{R}_1, z_1)\, A_0(R_1)\, T(R_1)\, P_2(\mathbf{R}_1 - \mathbf{R}_0, z_0), \qquad (15)$$

where

$$P_2(\mathbf{R}) = \frac{1}{i\lambda z}\exp\left(i\pi\frac{R^2}{\lambda z}\right) = P(x, z)\,P(y, z). \qquad (16)$$

The limits of integration are infinite but they are restricted by the aperture function $A_0(R) = \theta(R_a^2 - R^2)$.

In paper [2] the radius of the effective aperture was obtained from the maximum intensity value at the focus when $\mathbf{R} = \mathbf{R}_0 = 0$ and $z_0^{-1} + z_1^{-1} = f^{-1}$. In this case we have, using circular coordinates,

$$E(0) = -I_{02}^{1/2}\frac{2\pi}{\lambda z_1}\int_0^{R_a} \mathrm{d}R\, R \exp\left(-\pi\gamma\frac{R^2}{\lambda f}\right). \qquad (17)$$

Here, $I_{02} = I_0^2$, where $I_0 = (\lambda z_0)^{-1}$ is the intensity for the 1D case. If $\gamma = 0$ then we obtain $I_m = I_{02}(\pi R_a^2/\lambda z_1)^2$. In the general case, we can write $I_m = I_{02}(\pi R_{e2}^2/\lambda z_1)^2$ if

$$R_{e2} = R_a\left[\frac{1 - \exp(-\alpha_2 R_a^2)}{\alpha_2 R_a^2}\right]^{1/2}, \quad \alpha_2 = \frac{\pi\gamma}{\lambda f}. \qquad (18)$$

In this paper we propose to determine the radius of the effective aperture from the integral intensity. It is easy to calculate the integral intensity just behind the CRL when $z_1 = 0$. In this case,

$$\int \mathrm{d}\mathbf{R}\, I(R) = I_{02}2\pi\int_0^{R_a} \mathrm{d}R\, R\exp\left(-2\pi\gamma\frac{R^2}{\lambda f}\right) = I_{02}\pi R_e^2, \qquad (19)$$

where

$$R_e = R_a\left[\frac{1 - \exp(-\alpha R_a^2)}{\alpha R_a^2}\right]^{1/2}, \quad \alpha = \frac{2\pi\gamma}{\lambda f}. \qquad (20)$$

In the limit of the strongly absorbing CRL we find from this formula that the diameter of the effective aperture $D_e = 2R_e = 2\alpha^{-1/2} = (2\lambda f/\pi\gamma)^{1/2}$. We can compare this value with the aperture A_e of the linear (1D) CRL and obtain the relation $A_e = \pi^{1/2}R_e = (\pi^{1/2}/2)D_e = 0.8862D_e$. It is of interest that this relation means $A_e^2 = \pi R_e^2$. Mathematically, this relation is necessary because for a strongly absorbing CRL the integral intensity of the 2D CRL is a product of the integral intensities of the 1D CRL for the x and y coordinates independently.

4. Analytical account of the effective aperture

We note that the transmission function of a circular (2D) CRL, equation (14), can be written as $T(R) = T(x)T(y)$. The Fresnel propagator (16) has the same property. This is not the case for the function $A_0(R)$. This function does not allow us to divide the integral over $\mathbf{R}$ into the product of two integrals over x and y separately.

To obtain the possibility of analytical calculations we propose to replace the function $A_0(R)$ in equation (15) by the function

$$A_2(R) = \exp\left(-\frac{x^2 + y^2}{2R_a^2}\right), \qquad (21)$$

which gives the same integral intensity as $A_0(R)$, *i.e.* πR_a^2, but in the infinite limits. As for the FWHM D of the function $A_2^2(R)$ in any direction, we have $D = 2(\ln 2)^{1/2} R_a = 1.665 R_a = 0.8326 D_a$, where $D_a = 2R_a$. We see that D is less than D_a by 17%. On the other hand, $A_2^2(R_a) = A_2^2(0)/e$.

The main advantage of the function $A_2(R)$ is the property $A_2(R) = A(x)A(y)$, where $A(x) = \exp(-x^2/2R_a^2)$. The replacement of A_0 by A_2 in (15) allows us to replace the 2D integral by the product of two 1D integrals, namely $E(\mathbf{R}) = E(x)E(y)$ and

$$E(x) = \int \mathrm{d}x_1 \, P(x - x_1, z_1)\, A(x_1)\, T(x_1)\, P(x_1 - x_0, z_0). \qquad (22)$$

We note that in paper [1] the finite angular divergence of the incident beam was taken into account. To simplify the calculation we assume here that the angular divergence leads to the incident beam size at the CRL which is larger than the CRL aperture.

Now the integral can be calculated analytically and we obtain

$$E(x) = (i\lambda z_0)^{-1/2} C F(x, x_0) \exp\left[-i\pi \frac{(x + x_0 z_1/z_0)^2}{\lambda(z_2 + i z_3)}\right], \qquad (23)$$

where

$$C = \left(\frac{z_1}{z_2 + i z_3}\right)^{1/2}, \quad F(x, x_0) = \exp\left[i\frac{\pi}{\lambda}\left(\frac{x^2}{z_1} + \frac{x_0^2}{z_0}\right)\right]. \qquad (24)$$

Here,

$$z_2 = z_1 - z_1^2\left(\frac{1}{f} - \frac{1}{z_0}\right), \quad z_3 = z_1^2\left(\frac{\lambda}{2\pi R_a^2} + \frac{\gamma}{f}\right). \qquad (25)$$

The formulae (23)–(25) allow us to consider the intensity profile at any distance z_1. Let us consider $z_1 = 0$, *i.e.* just behind the CRL. In this case we need to account for an imaginary part accurately by means of considering very small z_1 values. As a result we obtain

$$E(x) = P(x_0) \exp\left(-\frac{1}{2}\alpha x^2\right), \quad \alpha = \frac{1}{R_a^2} + \frac{2\pi\gamma}{\lambda f}. \qquad (26)$$

The radius R_e of the effective aperture of the circular (2D) CRL by definition is determined from the 2D integral intensity which is presented as $I_{02}\pi R_e^2$. We note that the 2D integral is the square of the 1D integral and is equal to $I_{02}\pi\alpha^{-1}$. A calculation in the circular coordinates gives the same result. Finally we obtain

$$R_e = \alpha^{-1/2} = R_a\left(1 + \frac{2\pi\gamma}{\lambda f} R_a^2\right)^{-1/2}. \qquad (27)$$

This is another form of the interpolation formula for the radius of the effective aperture which follows from the approximate analytical approach.

Fig. 2 shows a comparison of two formulae for the case of the 2D CRL made from Al, $R = 200$ µm, $N = 20$, $\lambda = 0.1$ nm and for various values of R_a from 3 to 303 µm. One can see that for small values of R_a from 20 to 120 µm when absorption is not complete at the edge of aperture the approximate formula (27) (black curve) gives smaller values than the accurate formula (20) (red curve) but the difference is not more than 13%. For large values of R_a both curves coincide.

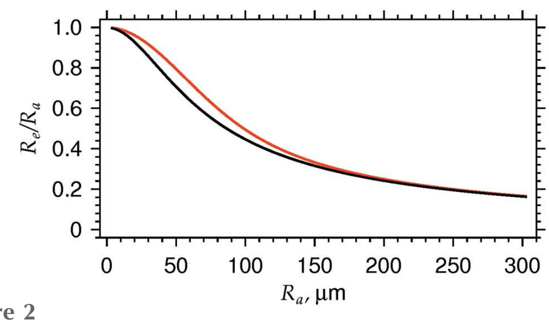


Figure 2
The dependence of the effective aperture of the 2D CRL R_e from the real aperture R_a for the case of an aluminium CRL with a curvature radius R = 200 µm, number of lenses N = 20 and wavelength λ = 0.1 nm, calculated by accurate formula (red curve) and approximate formula (black curve).

The proposed approximation allows one to simplify calculations because it is sufficient to calculate only the 1D CRL. Then the intensity profile of the 2D CRL can be obtained from the 1D CRL intensity profile as a product of two profiles for the x and y coordinates.

Let us consider now the distance of imaging the point source $z_1 = f(1 - f/z_0)^{-1}$, when $z_2 = 0$. It is sufficient to write $x_0 = 0$ because the dependence on x_0 is evident. Correspondingly for the 1D intensity $I(x)$ and 2D intensity $I_2(x, y)$ we have

$$I(x) = I_0 \frac{z_1}{z_3} \exp\left(-2\pi \frac{x^2}{\lambda z_3}\right), \quad I_2(x, y) = I(x)\, I(y), \qquad (28)$$

and for the integral intensities we obtain

$$\int \mathrm{d}x\, I(x) = I_0 \frac{\lambda^{1/2} z_1}{2^{1/2} z_3^{1/2}} = I_0 A_e, \qquad (29)$$

$$\int \mathrm{d}x\, \mathrm{d}y\, I_2(x, y) = I_{02} \frac{\lambda z_1^2}{2 z_3} = I_{02}\pi R_e^2. \qquad (30)$$

Here we have used $A_e = \pi^{1/2} R_e$.

This means that the approximation gives accurate formulae for the integral intensities at the focus but with R_e determined by (27). The maximum values of I_m and FWHM w_m for the 1D CRL are described by equation (11). For the 2D CRL the maximum value equals I_m^2 and w_m stays the same.

We note that, in the works of Kohn (2009, 2012), recurrent relations were derived for calculating the systems of many 1D CRLs. The analytical account of the aperture proposed in this section allows one to use these recurrent relations for systems of 2D CRLs too.

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